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On the Nonlinear Mean Ergodic Theorems for Asymptotically Nonexpansive Mappings in Banach Spaces

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1. Introduction.

Throughout this note X denotes a uniformly convex real Banach space and C is a closed convex subset of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*).

The duality mapping J (multi-valued) from X into X^* will be defined by $J(x) = \{x^* \in X^*: (x, x^*) = \|x\|^2 = \|x^*\|^2\}$ for $x \in X$.

We say that X is (F) if the norm of X is Fréchet differentiable, i.e., for each $x \in X$ with $x \neq 0$, $\lim_{t\to 0} t^{-1}(\|x+ty\|-\|x\|)$ exists uniformly in $y \in B_1$, where $B_r = \{z \in X : \|z\| \leq r\}$ for $r > 0$. It is easily seen that X is (F) if and only if for any bounded set $B \subset X$ and any $x \in X$, $\lim_{t\to 0}(2t)^{-1}(\|x+ty\|^2-\|x\|^2) = (y, J(x))$ uniformly in $y \in B$. We say that X satisfies Opial's condition if $w\text{-}\lim_{n\to\infty} x_n = x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in X$ with $y \neq x$.

A mapping $T : C \to C$ is said to be asymptotically nonexpansive if for each $n = 1, 2, \cdots$

(1.1) $\|T^n x - T^n y\| \leq (1+\alpha_n)\|x-y\|$ for any $x, y \in C$,

where $\lim_{n\to\infty} \alpha_n = 0$. In particular, if $\alpha_n = 0$ for $n \geq 1$, T is said to be nonexpansive. The set of fixed points of T will be denoted by $F(T)$.

Throughout the rest of this note let $T : C \to C$ be an asymptotically nonexpansive mapping satisfying (1.1).

A sequence $\{x_n\}_{n\geq 0}$ in C is called an almost-orbit of T if

(1.2) $\lim_{n\to\infty} [\sup_{m\geq 0} \|x_{n+m} - T^m x_n\|] = 0.$

A sequence $\{z_n\}$ in X is said to be strongly (or weakly) almost convergent to $z \in X$ if $\frac{1}{n}\sum_{i=0}^{n-1} z_{i+k}$ converges strongly (or weakly) as $n \to \infty$ to z uniformly in $k \geq 0$. The convex hull of a set E ($\subset X$) is denoted by co E, the closed convex hull by clco E, and $\omega_w(\{x_n\})$ denotes the set of weak subsequential limits of $\{x_n\}$ as $n \to \infty$.

We get the following (nonlinear) mean ergodic theorems.

Theorem 1. Suppose that $\{x_n\}_{n\geq 0}$ is an almost-orbit of T and C is bounded. If X satisfies Opial's condition or if X is (F), then $\{x_n\}$ is weakly almost convergent to an element of F(T).

Theorem 2. Suppose that $\{x_n\}_{n\geq 0}$ is an almost-orbit of T and C is bounded. If $\lim_{n\to\infty} \|x_n - x_{n+i}\|$ exists uniformly in $i \geq 0$, then $\{x_n\}$ is strongly almost convergent to an element of F(T).

Theorem 1 is an extension of [5, Theorem 1.], [1, Corollary 2.1], [4, Theorem 2.1] and Theorem 2 is an extension of [6, Theorem 1].

2. Lemmas.

Throughout this section, we assume that C is bounded. By Bruck's inequality [2, Theorem 2.1], we get

Lemma 1. There exists a strictly increasing, continuous, convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0)=0$ such that

$$\|T^k(\sum_{i=1}^{n}\lambda_i x_i) - \sum_{i=1}^{n}\lambda_i T^k x_i\|$$

$$\leq (1+\alpha_k)\gamma^{-1}(\max_{1\leq i,j\leq n}[\|x_i - x_j\| - \frac{1}{1+\alpha_k}\|T^k x_i - T^k x_j\|])$$

for any k, $n \geq 1$, any $\lambda_1, \cdots, \lambda_n \geq 0$ with $\sum_{i=1}^{n}\lambda_i = 1$, and any $x_1, \cdots, x_n \in C$.

Hereafter, let γ be as in Lemma 1.

Lemma 2. Suppose that $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are almost-orbits of T. Then $\{\|x_n - y_n\|\}$ converges as $n \to \infty$.

Proof. Put $a_n = \sup_{m\geq 0}\|x_{n+m} - T^m x_n\|$ and $b_n = \sup_{m\geq 0}\|y_{n+m} - T^m y_n\|$ for $n \geq 0$. Then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.
Since

$$\|x_{n+m} - y_{n+m}\| \leq \|x_{n+m} - T^m x_n\| + \|T^m x_n - T^m y_n\| + \|T^m y_n - y_{n+m}\|$$

$\leq a_n + b_n + (1+\alpha_m)\|x_n - y_n\|$, we have

$$\limsup_{m\to\infty}\|x_m - y_m\| \leq a_n + b_n + \|x_n - y_n\| \text{ for every } n \geq 0.$$

Taking the lim inf as $n \to \infty$,
we obtain $\limsup_{m\to\infty}\|x_m - y_m\| \leq \liminf_{n\to\infty}\|x_n - y_n\|$ and so the conclusion holds. Q.E.D.

We now put D = diameter C and $M = \sup_{n\geq 1}(1+\alpha_n)$.

Lemma 3. Suppose that $\{x_j^{(p)}\}_{j\geq 1}$ $(p = 1, 2, \cdots)$ are almost-orbits of T. Then for any $\varepsilon > 0$ and $n \geq 1$ there exist $N_\varepsilon \geq 1$ and $i_n(\varepsilon) \geq 1$, where N_ε is independent of n, such that

$$\|T^k(\sum_{p=1}^{n} \lambda_p x_i^{(p)}) - \sum_{p=1}^{n} \lambda_p T^k x_i^{(p)}\| < \varepsilon \text{ for any } k \geq N_\varepsilon, \text{ any } i \geq i_n(\varepsilon),$$

and any $\lambda_1, \cdots, \lambda_n \geq 0$ with $\sum_{p=1}^{n} \lambda_p = 1$.

Proof. For any $\varepsilon > 0$ choose $\delta > 0$ so that $\gamma^{-1}(\delta) < \varepsilon/M$. Then there exists $N_\varepsilon \geq 1$ such that $\alpha_k < \delta/4D$ for $k \geq N_\varepsilon$.
Since $\{\|x_j^{(p)} - x_j^{(q)}\|\}_{j\geq 1}$ converges as $j \to \infty$ by Lemma 2, for each $p, q \geq 1$ there exists $i_0(\varepsilon, p, q) \geq 1$ such that $\|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| < \delta/4$ if $i \geq i_0(\varepsilon, p, q)$ and $k \geq 0$.
Moreover, there is $i_1(\varepsilon, p) \geq 1$ such that $a_i^{(p)} < \delta/4$ for all $i \geq i_1(\varepsilon, p)$, where $a_i^{(p)} = \sup_{j\geq 0} \|x_{i+j}^{(p)} - T^j x_i^{(p)}\|$.
Put $i_n(\varepsilon) = \max\{i_0(\varepsilon, p, q),\ i_1(\varepsilon, p) : 1 \leq p, q \leq n\}$ for $n \geq 1$.
If $i \geq i_n(\varepsilon)$ and $k \geq N_\varepsilon$, then

$$\|x_i^{(p)} - x_i^{(q)}\| - \frac{1}{1+\alpha_k}\|T^k x_i^{(p)} - T^k x_i^{(q)}\|$$

$$\leq \|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| + a_i^{(p)} + a_i^{(q)} + \alpha_k\|x_i^{(p)} - x_i^{(q)}\| < \delta$$

for $1 \leq p, q \leq n$ and by Lemma 1,

$$\|T^k(\sum_{p=1}^{n} \lambda_p x_i^{(p)}) - \sum_{p=1}^{n} \lambda_p T^k x_i^{(p)}\| < \varepsilon$$

for any $\lambda_1, \cdots, \lambda_n \geq 0$ with $\sum_{p=1}^{n} \lambda_p = 1$. Q. E. D.

For any $\varepsilon > 0$ and $k \geq 1$, we put $F_\varepsilon(T^k) = \{x \in C : \|T^k x - x\| \leq \varepsilon\}$. Since C is bounded, $F(T) \neq \Phi$. (For example, see [3, Theorem 1].)

Lemma 4. Suppose that $\{x_i\}_{i\geq 0}$ is an almost-orbit of T. Then for any $\varepsilon > 0$ there exists $N_\varepsilon \geq 1$ such that for each $k \geq N_\varepsilon$, there is $N_k(=N_k(\varepsilon)) \geq 1$ satisfying

$$\frac{1}{n}\sum_{i=0}^{n-1} x_{i+\ell} \in F_\varepsilon(T^k) \text{ for all } n \geq N_k \text{ and all } \ell \geq 0.$$

Proof. Let $\varepsilon > 0$ be arbitrarily given and σ be the inverse function of $t \mapsto M\gamma^{-1}(3t) + t$. Put $\delta = \min\{\sigma(\frac{\varepsilon}{3}), \frac{\varepsilon}{3MD}\}$ and $M' = M+1$. Choose $\eta > 0$ and $N_{1,\varepsilon} \geq 1$ so that $\gamma^{-1}(\eta) < \frac{\delta^2}{2M}$ and $\alpha_k < \sigma(\frac{\varepsilon}{3})/D$ for $k \geq N_{1,\varepsilon}$. Furthermore, by Lemma 3, there exists $N_{2,\varepsilon} \geq 1$ such that for any $p \geq 1$ there is $i_p(\varepsilon) \geq 1$ satisfying

$$(2.1)\qquad \|T^k(\frac{1}{p}\sum_{j=0}^{p-1} x_{i+j+\ell}) - \frac{1}{p}\sum_{j=0}^{p-1} T^k x_{i+j+\ell}\| < \delta^2/8$$

for any $k \geq N_{2,\varepsilon}$, any $i \geq i_p(\varepsilon)$, and any $\ell \geq 0$.

Put $N_\varepsilon = \max(N_{1,\varepsilon}, N_{2,\varepsilon})$ and let $k \geq N_\varepsilon$ be fixed. By Lemma 1 and the choice of δ, we get

$$(2.2)\qquad \mathrm{clco}\, F_\delta(T^k) \subset F_{\varepsilon/3}(T^k).$$

Next, choose $p \geq 1$ so that $\frac{Dk}{p} \leq \frac{\delta^2}{2}$ and let p be fixed. Since $\{x_i\}_{i\geq 0}$ is an almost-orbit of T, there exists $N \geq 1$ such that $\sup_{q\geq 0} \|x_{m+q} - T^q x_m\| < \frac{\delta^2}{8}$ for $m \geq N$. Set $w_i = \frac{1}{p}\sum_{j=0}^{p-1} x_{i+j}$ for $i \geq 0$. If $i \geq i_p(\varepsilon) + N$, by (2.1),

$$\|w_{i+k+\ell}-T^kw_{i+\ell}\|$$

$$\leq \Big\|\frac{1}{p}\sum_{j=0}^{p-1}(x_{i+j+k+\ell}-T^kx_{i+j+\ell})\Big\| + \Big\|\frac{1}{p}\sum_{j=0}^{p-1}T^kx_{i+j+\ell}-T^k(\frac{1}{p}\sum_{j=0}^{p-1}x_{i+j+\ell})\Big\| < \frac{\delta^2}{4}$$

for all $\ell \geq 0$. Choose $N_3(k) \geq i_p(\varepsilon)+N+1$ such that $\frac{D(i_p(\varepsilon)+N)}{n} < \frac{\delta^2}{4}$ for all $n \geq N_3(k)$. If $n \geq N_3(k)$, then

$$(2.3)\qquad \frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+\ell}-T^kw_{i+\ell}\| \leq \frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+\ell}-w_{i+k+\ell}\|$$

$$+\frac{1}{n}\Big(\sum_{i=0}^{i_p+N-1}+\sum_{i=i_p+N}^{n-1}\Big)\|w_{i+k+\ell}-T^kw_{i+\ell}\| \leq \frac{Dk}{p}+\frac{(i_p+N)D}{n}+\frac{\delta^2}{4} \leq \delta^2$$

for all $\ell \geq 0$, where $i_p = i_p(\varepsilon)$. Finally, choose $N_4(k) \geq 1$ so that $\frac{(p-1)D}{2n} < \frac{\varepsilon}{3M}$ for all $n \geq N_4(k)$. Put $N_k = \max(N_3(k), N_4(k))$ and let $n \geq N_k$ be fixed and $\ell \geq 0$.

Set $A(k,n,\ell) = \{i \in Z : 0 \leq i \leq n-1 \text{ and } \|w_{i+\ell}-T^kw_{i+\ell}\| \geq \delta\}$ and $B(k,n,\ell) = \{0,1,\cdots,n-1\}\setminus A(k,n,\ell)$. By (2.3), $\#A(k,n,\ell) \leq n\delta$, where # denotes cardinality. Let $f \in F(T)$. Then,

$$\frac{1}{n}\sum_{i=0}^{n-1}x_{i+\ell} = \frac{1}{n}\sum_{i=0}^{n-1}w_{i+\ell}+\frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i+\ell-1}-x_{i+\ell+n-1})$$

$$= \Big[\frac{1}{n}(\#A(k,n,\ell))\cdot f+\frac{1}{n}\sum_{i\in B(k,n,\ell)}w_{i+\ell}\Big]+\Big[\frac{1}{n}\sum_{i\in A(k,n,\ell)}(w_{i+\ell}-f)\Big]$$

$$+\frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i+\ell-1}-x_{i+\ell+n-1}).$$

The first term on the right side of the above equality is contained in $\mathrm{clco}\, F_\delta(T^k)$, and the rest term in $B_{2\varepsilon/3M}$. By (2.2), we get

$$\frac{1}{n}\sum_{i=0}^{n-1}x_{i+\ell} \in F_\varepsilon(T^k) \text{ for all } \ell \geq 0. \qquad \text{Q.E.D.}$$

Lemma 5. Let $\{x_n\}$ in C be such that $\text{w-}\lim_{n\to\infty} x_n = x$. Suppose that for any $\varepsilon > 0$ there exists $N(\varepsilon) \geq 1$ such that for $k \geq N(\varepsilon)$ there is $N_k \geq 1$ satisfying $\|T^k x_n - x_n\| < \varepsilon$ for all $n \geq N_k$. Then $x \in F(T)$.

Proof. We shall show that $\lim_{k\to\infty} \|T^k x - x\| = 0$. For any $\varepsilon > 0$ choose $\delta > 0$ so that $\gamma^{-1}(\delta) < \frac{\varepsilon}{4M}$ and take $N_1(\varepsilon) \geq 1$ such that $\alpha_k < \frac{\delta}{3D}$ for all $k \geq N_1(\varepsilon)$. Put $\delta' = \min\left(\frac{\delta}{3}, \frac{\varepsilon}{4}\right)$.
By the assumption, there exists $N(\varepsilon) \geq 1$ such that for each $k \geq N(\varepsilon)$ there is $N_k \geq 1$ satisfying $\|T^k x_n - x_n\| < \delta'$ for all $n \geq N_k$.
Put $N_2(\varepsilon) = \max(N_1(\varepsilon), N(\varepsilon))$ and let $k \geq N_2(\varepsilon)$ be arbitrarily fixed. Since $x \in \overline{co}\{x_n : n \geq N_k\}$, there exists a sequence

$\{\sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)}\} \subset co\{x_n : n \geq N_k\}$ such that $\lim_{n\to\infty} \sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)} = x$.

Therefore there is $N_3(k) \geq 1$ such that $\|\sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)} - x\| < \frac{\varepsilon}{4M}$ for all $n \geq N_3(k)$ and hence if $n \geq N_3(k)$, $\|T^k x - T^k(\sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)})\| < \frac{\varepsilon}{4}$.

On the other hand, by Lemma 1 and the choice of δ and k, we get

$$\|T^k(\sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)}) - \sum_{i=1}^{\ell_n} \lambda_n^{(i)} T^k x_{\psi_n(i)}\| < \frac{\varepsilon}{4} \text{ for all } n \geq 1.$$

Consequently, $\|T^k x - x\| \leq \|T^k x - T^k(\sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)})\|$

$$+ \|T^k(\sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)}) - \sum_{i=1}^{\ell_n} \lambda_n^{(i)} T^k x_{\psi_n(i)}\|$$

$$+ \| \sum_{i=1}^{\ell_n} \lambda_n^{(i)} (T^k x_{\psi_n(i)} - x_{\psi_n(i)}) \| + \| \sum_{i=1}^{\ell_n} \lambda_n^{(i)} x_{\psi_n(i)} - x \| < \varepsilon,$$

where $n \geq N_3(k)$.

This shows that $\|T^k x - x\| < \varepsilon$ for $k \geq N_2(\varepsilon)$. Q. E. D.

Lemma 6. Suppose that X is (F) and $\{x_n\}$ is an almost-orbit of T. Then the following hold:

(i) $\{(x_n, J(f-g))\}$ converges for every $f, g \in F(T)$.

(ii) $F(T) \cap \operatorname{clco} \omega_w(\{x_n\})$ is at most a singleton.

Proof. Let $\lambda \in (0,1)$ and $f, g \in F(T)$. By Lemma 3, for any $\varepsilon > 0$ there exist $N_\varepsilon \geq 1$ and $i_2(\varepsilon) \geq 1$ such that if $k \geq N_\varepsilon$ and $n \geq i_2(\varepsilon)$,

$$\|T^k(\lambda x_n + (1-\lambda) f) - \lambda T^k x_n - (1-\lambda) f\| < \varepsilon.$$

Since $\|\lambda x_{n+m} + (1-\lambda) f - g\| \leq \lambda \|x_{n+m} - T^m x_n\|$

$$+ \|T^m(\lambda x_n + (1-\lambda) f) - \lambda T^m x_n - (1-\lambda) f\| + (1+\alpha_m) \|\lambda x_n + (1-\lambda) f - g\|$$

$$\leq \sup_{\ell \geq 0} \|x_{n+\ell} - T^\ell x_n\| + \varepsilon + (1+\alpha_m) \|\lambda x_n + (1-\lambda) f - g\|$$

for $m \geq N_\varepsilon$ and $n \geq i_2(\varepsilon)$, we have

$$\limsup_{m\to\infty} \|\lambda x_m + (1-\lambda) f - g\| \leq \sup_{\ell \geq 0} \|x_{n+\ell} - T^\ell x_n\| + \varepsilon + \|\lambda x_n + (1-\lambda) f - g\|$$

for $n \geq i_2(\varepsilon)$. Letting $n \to \infty$ and then $\varepsilon \downarrow 0$, we get

$$\limsup_{m\to\infty} \|\lambda x_m + (1-\lambda) f - g\| \leq \liminf_{n\to\infty} \|\lambda x_n + (1-\lambda) f - g\|$$

and so $\|\lambda x_n + (1-\lambda) f - g\|$ converges as $n \to \infty$.

The boundedness of $\{\|x_n - f\|\}_{n\geq 0}$ and the Fréchet differentiability of X imply that $a(\lambda,n) = (2\lambda)^{-1}(\|f-g+\lambda(x_n - f)\|^2 - \|f-g\|^2)$ converges to $(x_n - f, J(f-g))$ as $\lambda \downarrow 0$ uniformly in $n \geq 0$.

Hence $\lim_{n\to\infty} (x_n - f, J(f-g)) = \lim_{\lambda\to 0+,\, n\to\infty} a(\lambda,n)$ exists. This proves (i).

It follows from (i) that $(u-v, J(f-g)) = 0$ for all $u, v \in \omega_w(\{x_n\})$ and hence for all $u, v \in \mathrm{clco}\ \omega_w(\{x_n\})$. Therefore, $F(T) \cap \mathrm{clco}\ \omega_w(\{x_n\})$ is at most a singleton. Q.E.D.

We set

$$s(n;m) = \frac{1}{n}\sum_{i=0}^{n-1} x_{i+m} \quad (n \geq 1 \ ; \ m \geq 0)$$

for an almost-orbit $\{x_n\}$ of T.

Lemma 7. Let $\{x_n\}$ be an almost-orbit of T. Then there exists a sequence $\{i_n\}$ of nonnegative integers with $i_n \to \infty$ as $n \to \infty$ satisfying the following:

Let $\{k_n\}$ be a sequence of nonnegative integers with $k_n \geq i_n$ for all n. Then, we have the following:

(i) $\|s(n;k_n) - f\|$ is convergent as $n \to \infty$ for every $f \in F(T)$.

(ii) If X satisfies Opial's condition or if X is (F), then there exists an element f of F(T) such that $w\text{-}\lim_{n\to\infty} s(n;k_n) = f$.

Moreover, $F(T) \cap \mathrm{clco}\ \omega_w(\{x_n\}) = \{f\}$ in case X is (F).

Proof. By Lemma 3, there exist divergent sequences $\{N_n\}$ and $\{i_n\}$ of nonnegative integers such that if $k \geq N_n$ and $i \geq i_n$,

$$\text{(2.4)} \qquad \left\|T^{k}\left(\frac{1}{n}\sum_{p=0}^{n-1} x_{p+i}\right) - \frac{1}{n}\sum_{p=0}^{n-1} T^{k}x_{p+i}\right\| < \frac{1}{n}.$$

Let $f \in F(T)$ and $\{k_n\}$ be a sequence of nonnegative integers with $k_n \geq i_n$ for all n. By (2.4),

$$\left\|\frac{1}{n+m}\left(\sum_{p=0}^{k_n+N_n-1} + \sum_{p=k_n+N_n}^{n+m-1}\right)\left(\frac{1}{n}\sum_{q=0}^{n-1} x_{p+q+k_{n+m}} - f\right)\right\|$$

$$\leq \frac{(k_n+N_n)D}{n+m} + \frac{1}{n+m}\sum_{p=k_n+N_n}^{n+m-1}\left\|\frac{1}{n}\sum_{q=0}^{n-1}\left(x_{p+q+k_{n+m}} - T^{p+k_{n+m}-k_n}x_{q+k_n}\right)\right.$$

$$+\left(\frac{1}{n}\sum_{q=0}^{n-1} T^{p+k_{n+m}-k_n}x_{q+k_n} - T^{p+k_{n+m}-k_n}\left(\frac{1}{n}\sum_{q=0}^{n-1} x_{q+k_n}\right)\right)$$

$$\left. + \left(T^{p+k_{n+m}-k_n}\left(\frac{1}{n}\sum_{q=0}^{n-1} x_{q+k_n}\right) - f\right)\right\|$$

$$\leq \frac{(k_n+N_n)D}{n+m} + \frac{1}{n}\sum_{q=0}^{n-1}\sup_{\ell\geq 0}\|x_{\ell+q+k_n} - T^{\ell}x_{q+k_n}\| + \frac{1}{n} + \|s(n;k_n) - f\|$$

$$+ \frac{1}{n+m}\sum_{p=k_n+N_n}^{n+m-1}\alpha_{p+k_{n+m}-k_n} D \quad \text{whenever } n + m \geq k_n + N_n + 1.$$

Therefore,

$$\|s(n+m;k_{n+m}) - f\|$$

$$\leq \left\|\frac{1}{n+m}\left(\sum_{p=0}^{k_n+N_n-1} + \sum_{p=k_n+N_n}^{n+m-1}\right)\left(\frac{1}{n}\sum_{q=0}^{n-1} x_{p+q+k_{n+m}} - f\right)\right\|$$

$$+ \frac{1}{n(n+m)}\sum_{p=1}^{n-1}(n-p)\|x_{p+k_{n+m}-1} - x_{p+k_{n+m}+n+m-1}\|$$

$$\leq \frac{(k_n+N_n)D}{n+m} + \frac{1}{n}\sum_{q=0}^{n-1}\sup_{\ell\geq 0}\|x_{\ell+q+k_n} - T^{\ell}x_{q+k_n}\| + \frac{1}{n} + \|s(n;k_n) - f\|$$

$$+\frac{1}{n+m}\sum_{p=k_n+N_n}^{n+m-1}\alpha_{p+k_{n+m}-k_n}D+\frac{(n-1)D}{2(n+m)}\quad \text{for } n+m \geq k_n+N_n+1.$$

Hence

$$\limsup_{m\to\infty}\|s(m;k_m)-f\| \leq \liminf_{n\to\infty}\|s(n;k_n)-f\|.$$

This proves (i).

Now, let W be the set of weak subsequential limits of $\{s(n;k_n)\}$ as $n \to \infty$. Since X is reflexive and $\{s(n;k_n)\}$ is bounded, W is nonempty. To prove (ii) it suffices to show that $W \subset F(T)$ and W is a singleton. By Lemmas 4 and 5, $W \subset F(T)$ and so $\{\|s(n;k_n)-v\|\}$ converges as $n \to \infty$ for every $v \in W$ by (i).

First, suppose that X satisfies Opial's condition and let $v_i \in W$, $i = 1,2$ and $v_i = \text{w-}\lim_{n(i)\to\infty} s(n(i);k_{n(i)})$, where $\{n(i)\}$, $i = 1,2$, are subsequences of $\{n\}$. Suppose $v_1 \neq v_2$. Then, by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty}\|s(n;k_n)-v_1\| &= \lim_{n(1)\to\infty}\|s(n(1);k_{n(1)})-v_1\| \\ &< \lim_{n(1)\to\infty}\|s(n(1);k_{n(1)})-v_2\| \\ &= \lim_{n\to\infty}\|s(n;k_n)-v_2\|.\end{aligned}$$

In the same way we have $\lim_{n\to\infty}\|s(n;k_n)-v_2\| < \lim_{n\to\infty}\|s(n;k_n)-v_1\|$. This is a contradiction. Consequently, $v_1 = v_2$ and W is a singleton.

Next, suppose that X is (F). We can easily see that

$$W \subset \bigcap_{i=0}^{\infty} \text{clco}\,\{x_n : n \geq i\} = \text{clco}\,\omega_w(\{x_n\}).$$

Thus $W \subset F(T) \cap \text{clco}\,\omega_w(\{x_n\})$ and hence W is a singleton by Lemma 6 (ii).

Q. E. D.

Lemma 8. Let $\{x_n\}$ be an almost-orbit of T and $\{k_n\}$ a sequence of nonnegative integers. If $\{s(n;k_n+\ell)\}$ converges weakly (or strongly) as $n \to \infty$, uniformly in $\ell \geq 0$, to an element y of X, then $\{s(n;\ell)\}$ converges weakly (or strongly) as $n \to \infty$, uniformly in $\ell \geq 0$, to y.

Proof. Suppose that $\lim_{n\to\infty} s(n;k_n+\ell) = y$ uniformly in $\ell \geq 0$. Then, for any $\varepsilon > 0$ there is $N \geq 1$ such that $\|s(N;k_N+\ell) - y\| < \varepsilon$ for all $\ell \geq 0$.

$$\|s(n;\ell) - y\| \leq \frac{1}{n}\Big(\sum_{i=0}^{k_N-1} + \sum_{i=k_N}^{n-1}\Big)\|s(N;i+\ell) - y\|$$
$$+ \frac{1}{nN}\sum_{i=1}^{N-1}(N-i)\|x_{i+\ell-1} - x_{i+\ell+n-1}\|$$
$$\leq \frac{k_N D}{n} + \varepsilon + \frac{(N-1)D}{2n} \text{ for } n \geq k_N + 1 \text{ and } \ell \geq 0.$$

This shows that $\lim_{n\to\infty} s(n;\ell) = y$ uniformly in $\ell \geq 0$.
In a similar way we can prove the weak case. Q.E.D.

Throughout the rest of this section, we assume that $\{x_n\}$ is an almost-orbit of T satisfying

(2.5) $\lim_{n\to\infty} \|x_n - x_{n+i}\|$ exists uniformly in $i \geq 0$.

Lemma 9. The following holds:

$$\lim_{\ell,m,n\to\infty} \Big\|T^\ell\Big(\frac{1}{2n}\sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} x_{i+m}\Big) - \Big(\frac{1}{2n}\sum_{i=0}^{n-1} T^\ell x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} T^\ell x_{i+m}\Big)\Big\| = 0.$$

In particular, $\lim_{\ell, n\to\infty} \|T^{\ell}(\frac{1}{n}\sum_{i=0}^{n-1} x_{i+n}) - \frac{1}{n}\sum_{i=0}^{n-1} T^{\ell}x_{i+n}\| = 0.$

Proof. By Lemma 1,

$$(2.6)\quad \|T^{\ell}(\frac{1}{2n}\sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} x_{i+m}) - (\frac{1}{2n}\sum_{i=0}^{n-1} T^{\ell}x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} T^{\ell}x_{i+m})\|$$

$$\le M\gamma^{-1}(\max\{\|x_{i+n} - x_{j+n}\| - \frac{1}{1+\alpha_\ell}\|T^{\ell}x_{i+n} - T^{\ell}x_{j+n}\|,\ \|x_{i+n} - x_{p+m}\|$$

$$- \frac{1}{1+\alpha_\ell}\|T^{\ell}x_{i+n} - T^{\ell}x_{p+m}\|,\ \|x_{p+m} - x_{q+m}\| - \frac{1}{1+\alpha_\ell}\|T^{\ell}x_{p+m} - T^{\ell}x_{q+m}\| :$$

$$0 \le i,j \le n-1,\ 0 \le p,q \le m-1\})\quad \text{for any } n,m \ge 1 \text{ and } \ell \ge 0.$$

For any $\varepsilon > 0$ choose $\delta > 0$ such that $\gamma^{-1}(\delta) < \varepsilon/M$. By the assumption, there exists $N \ge 1$ such that $\sup_{i\ge 0} |\ \|x_n - x_{n+i}\| - \|x_m - x_{m+i}\|\ | < \delta/4$, $\sup_{r\ge 0} \|x_{n+r} - T^r x_n\| < \delta/4$, and $\alpha_\ell < \delta/4D$ for every ℓ, m, $n \ge N$.

If ℓ, m, $n \ge N$, $\|x_{i+n} - x_{j+m}\| - \frac{1}{1+\alpha_\ell}\|T^{\ell}x_{i+n} - T^{\ell}x_{j+m}\|$

$$\le \|x_{i+n} - x_{j+m}\| - \|x_{i+\ell+n} - x_{j+\ell+m}\| + \|x_{i+\ell+n} - T^{\ell}x_{i+n}\|$$

$$+ \|x_{j+\ell+m} - T^{\ell}x_{j+m}\| + \alpha_\ell\|x_{i+n} - x_{j+m}\| < \delta \text{ for every } i,\ j \ge 0.$$

Combining this with (2.6),

$$\|T^{\ell}(\frac{1}{2n}\sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} x_{i+m}) - (\frac{1}{2n}\sum_{i=0}^{n-1} T^{\ell}x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} T^{\ell}x_{i+m})\| < \varepsilon$$

for every ℓ, m, $n \ge N$. Q.E.D.

Lemma 10. $\{s(n;n)\}$ is strongly convergent as $n \to \infty$ to an element y of $F(T)$.

Proof. Take $f \in F(T)$ and set $u_n = s(n;n) - f$ for $n \geq 1$. Similarly as the proof of Lemma 7 (i), using Lemma 9, we can see that $\|u_n\| = \|s(n;n) - f\|$ converges as $n \to \infty$. Put $d = \lim_{n\to\infty} \|u_n\|$. Then, we have

$$\lim_{n\to\infty} \|u_n + u_{n+i}\| = 2d \text{ for every } i \geq 1 \tag{2.7}$$

because $\|u_n - u_{n+1}\| \to 0$ as $n \to \infty$.

Since

$$s(n+k;n+k) = \frac{1}{n+k} \sum_{i=0}^{n+k-1} s(n;n+k+i) + v(n,k), \quad \|v(n,k)\| \leq \frac{(n-1)D}{2(n+k)},$$

where $v(n,k) = \frac{1}{n(n+k)} \sum_{i=1}^{n-1} (n-i)(x_{i+n+k-1} - x_{i+2(n+k)-1})$,

it follows that

$$\|u_{n+k} + u_{m+k}\| \leq \left\|\frac{1}{n+k} \sum_{i=0}^{n+k-1} (s(n;n+k+i) + s(m;m+k+i) - 2f)\right\|$$

$$+ \left\|\frac{m-n}{(m+k)(n+k)} \sum_{i=0}^{n+k-1} (s(m;m+k+i) - f)\right\|$$

$$+ \left\|\frac{1}{m+k} \sum_{i=n+k}^{m+k-1} (s(m;m+k+i) - f)\right\| + \|v(n,k)\| + \|v(m,k)\|$$

$$\leq \frac{2}{n+k} \sum_{i=0}^{n+k-1} \|2^{-1}(s(n;n+k+i) + s(m;m+k+i)) - f\| + \frac{2(m-n)D}{m+k}$$

$$+ \frac{(n-1)D}{2(n+k)} + \frac{(m-1)D}{2(m+k)} \text{ for } m \geq n \geq 1 \text{ and } k \geq 0.$$

Moreover,

$$\|2^{-1}(s(n;n+k+i) + s(m;m+k+i)) - f\|$$

$$\le \frac{1}{2n}\sum_{j=0}^{n-1} \sup_{\ell\ge 0} \|x_{j+n+\ell} - T^{\ell}x_{j+n}\| + \frac{1}{2m}\sum_{j=0}^{m-1} \sup_{\ell\ge 0} \|x_{j+m+\ell} - T^{\ell}x_{j+m}\|$$

$$+ \|(\frac{1}{2n}\sum_{j=0}^{n-1} T^{i+k}x_{j+n} + \frac{1}{2m}\sum_{j=0}^{m-1} T^{i+k}x_{j+m}) - T^{i+k}(\frac{1}{2n}\sum_{j=0}^{n-1} x_{j+n} + \frac{1}{2m}\sum_{j=0}^{m-1} x_{j+m})\|$$

$$+ (1 + \alpha_{i+k})\|2^{-1}s(n;n) + 2^{-1}s(m;m) - f\|$$

for $m,\ n \ge 1$ and $i,\ k \ge 0$.

By Lemma 9, for any $\varepsilon > 0$ there exists $N \ge 1$ such that

$$\|T^k(\frac{1}{2n}\sum_{i=0}^{n-1} x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} x_{i+m}) - (\frac{1}{2n}\sum_{i=0}^{n-1} T^k x_{i+n} + \frac{1}{2m}\sum_{i=0}^{m-1} T^k x_{i+m})\| < \varepsilon,$$

$\sup_{r\ge 0} \|x_{n+r} - T^r x_n\| < \varepsilon$, and $\alpha_k < \varepsilon/D$ for every $k,\ m,\ n \ge N$.

Consequently, we obtain

$$\|u_{n+k} + u_{m+k}\| \le 6\varepsilon + \|u_n + u_m\| + \frac{2(m-n)D}{m+k} + \frac{(n-1)D}{2(n+k)} + \frac{(m-1)D}{2(m+k)}$$

for every $m \ge n \ge N$ and $k \ge N$. Letting $k \to \infty$, it follows from (2.7) that $2d \le 6\varepsilon + \|u_n + u_m\|$ for every $m,\ n \ge N$. Hence

$$2d \le \liminf_{n,m\to\infty} \|u_n + u_m\| \le \limsup_{n,m\to\infty} \|u_n + u_m\| \le 2d$$

and so $\lim_{n,m\to\infty} \|u_n + u_m\| = 2d$. By uniform convexity of X and $\lim_{n\to\infty} \|u_n\| = d$, $\lim_{m\to\infty} \|s(n;n) - s(m;m)\| = \lim_{n,m\to\infty} \|u_n - u_m\| = 0$, whence $\{s(n;n)\}$ converges strongly. Put $y = \lim_{n\to\infty} s(n;n)$.

Then we have

$$\|y - T^{\ell}y\| \le \|y - s(n;n)\| + \|s(n;n) - s(n;n+\ell)\|$$

$$+ \left\|\frac{1}{n}\sum_{i=0}^{n-1}(x_{i+n+\ell} - T^{\ell}x_{i+n})\right\| + \left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{\ell}x_{i+n} - T^{\ell}\left(\frac{1}{n}\sum_{i=0}^{n-1}x_{i+n}\right)\right\|$$

$$+ \|T^{\ell}s(n;n) - T^{\ell}y\|$$

$$\leq (M + 1)\|y - s(n;n)\| + 2\varepsilon + \frac{\ell}{n}D \quad \text{for all } n,\ \ell \geq N.$$

Hence $\lim_{\ell\to\infty} \|T^{\ell}y - y\| = 0$ and so $y \in F(T)$. Q.E.D.

3. Proof of Theorems.

Proof of Theorem 1. Let $\{x_n\}$ be an almost-orbit of T. First, suppose that X is (F). By Lemma 7 (ii), there exist a sequence $\{i_n\}$ of nonnegative integers and an element y of F(T) such that $\{y\} = F(T) \cap \text{clco}\ \omega_w(\{x_n\})$ and $\text{w-}\lim_{n\to\infty} s(n;k_n) = y$ for any sequence $\{k_n\}$ with $k_n \geq i_n$ for all n. This implies that $\text{w-}\lim_{n\to\infty} s(n;i_n+\ell) = y$ uniformly in $\ell \geq 0$. Hence $\{x_n\}$ is weakly almost convergent to y by Lemma 8.

Next, suppose that X satisfies Opial's condition. We denote by Λ the set of sequences $\{k_n\}$ of nonnegative integers with $k_n \geq i_n$ for all n, where $\{i_n\}$ is as in Lemma 7. It follows from Lemma 7 (ii) that $\|s(n;k_n) - f\|$ converges as $n \to \infty$ for every $\{k_n\} \in \Lambda$ and $f \in F(T)$. Define $r(\{k_n\};f)$, $r(\{k_n\})$, and r by

$$r(\{k_n\};f) = \lim_{n\to\infty} \|s(n;k_n) - f\| \quad \text{for } \{k_n\} \in \Lambda \text{ and } f \in F(T),$$

$$r(\{k_n\}) = \inf\ \{r(\{k_n\};f) : f \in F(T)\} \quad \text{for } \{k_n\} \in \Lambda,$$

and

$$r = \inf\ \{r(\{k_n\}) : \{k_n\} \in \Lambda\},$$

respectively. Now, choose $\{k_n^{(i)}\} \in \Lambda$, $i = 1, 2, \cdots$, such that

$\lim_{i\to\infty} r(\{k_n^{(i)}\}) = r$, and let $h_n = \max\{k_n^{(i)} : 1 \leq i \leq n\} + N_n$ for $n \geq 1$, where $\{N_n\}$ is as in the proof of Lemma 7. Clearly $\{h_n\} \in \Lambda$. Moreover, we obtain

$$(3.1) \qquad r(\{h_n\}) = r.$$

To show this, let $n \geq i \geq 1$ and $f \in F(T)$. Then,

$$(3.2) \qquad \|s(n;h_n) - f\| \leq \frac{1}{n}\sum_{j=0}^{n-1} \|x_{j+h_n} - T^{h_n-k_n^{(i)}} x_{j+k_n^{(i)}}\|$$

$$+ \|\frac{1}{n}\sum_{j=0}^{n-1} T^{h_n-k_n^{(i)}} x_{j+k_n^{(i)}} - T^{h_n-k_n^{(i)}} (\frac{1}{n}\sum_{j=0}^{n-1} x_{j+k_n^{(i)}})\|$$

$$+ \|T^{h_n-k_n^{(i)}} (\frac{1}{n}\sum_{j=0}^{n-1} x_{j+k_n^{(i)}}) - f\| \leq \frac{1}{n}\sum_{j=0}^{n-1} \sup_{\ell\geq 0} \|x_{j+k_n^{(i)}+\ell} - T^{\ell} x_{j+k_n^{(i)}}\| + \frac{1}{n}$$

$$+ (1 + \alpha_{h_n-k_n^{(i)}})\|s(n;k_n^{(i)}) - f\|.$$

Letting $n \to \infty$, it follows that $r(\{h_n\};f) \leq r(\{k_n^{(i)}\};f)$ for all $f \in F(T)$ and so $r(\{h_n\}) \leq \lim_{i\to\infty} r(\{k_n^{(i)}\}) = r$.
But $r \leq r(\{h_n\})$ by the definition of r. Thus (3.1) holds.
Since $F(T)$ is closed convex (For example, see [3, Theorem 2].) and $\{s(n;h_n)\}$ is bounded, the reflexivity of X implies that there is an element y of $F(T)$ such that $r(\{h_n\};y) = r(\{h_n\})$ ($= r$).
Set $h_n' = h_n + N_n$. Then we shall show

$$(3.3) \qquad \text{w-}\lim_{n\to\infty} s(n;h_n'+\ell) = y \text{ uniformly in } \ell \geq 0.$$

If this is shown, the conclusion follows from Lemma 8.

To show (3.3) let $\{\ell_n\}$ be an arbitrary sequence such that $\ell_n \geq h'_n$ for all n. $\{\ell_n\} \in \Lambda$ and by Lemma 7 (ii) there exists $z \in F(T)$ such that $w\text{-}\lim_{n\to\infty} s(n;\ell_n) = z$. Suppose $z \neq y$.
Then Opial's condition implies that

$$r(\{\ell_n\}) \leq \lim_{n\to\infty} \|s(n;\ell_n) - z\| < \lim_{n\to\infty} \|s(n;\ell_n) - y\| = r(\{\ell_n\};y).$$

But, by the same way as in (3.2), we have
$r(\{\ell_n\};y) \leq r(\{h_n\};y) \leq r(\{h_n\}) = r$. Thus $r(\{\ell_n\}) < r$ and this contradicts the definition of r. Hence $z = y$ and so $w\text{-}\lim_{n\to\infty} s(n;\ell_n) = y$.
Clearly, this implies (3.3). Q.E.D.

Proof of Theorem 2. Let $\{x_n\}$ be an almost-orbit of T and suppose that $\lim_{n\to\infty} \|x_n - x_{n+i}\|$ exists uniformly in $i \geq 0$.
We shall show that there exists an element y of F(T) such that $\lim_{n\to\infty} s(n;2n+\ell) = y$ uniformly in $\ell \geq 0$. By Lemma 9, for any $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\left\|\frac{1}{n}\sum_{i=0}^{n-1} T^{n+\ell}x_{i+n} - T^{n+\ell}\left(\frac{1}{n}\sum_{i=0}^{n-1} x_{i+n}\right)\right\| < \varepsilon \text{ and } \sup_{r\geq 0} \|x_{n+r} - T^r x_n\| < \varepsilon$$

for every $n \geq N$ and $\ell \geq 0$.
By Lemma 10, there exists an element y of F(T) such that $\lim_{n\to\infty} s(n;n) = y$. Then we have

$$\|s(n;2n+\ell) - y\| \leq \frac{1}{n}\sum_{i=0}^{n-1} \|x_{i+2n+\ell} - T^{n+\ell}x_{i+n}\|$$

$$+ \left\|\frac{1}{n}\sum_{i=0}^{n-1} T^{n+\ell}x_{i+n} - T^{n+\ell}\left(\frac{1}{n}\sum_{i=0}^{n-1} x_{i+n}\right)\right\| + \left\|T^{n+\ell}\left(\frac{1}{n}\sum_{i=0}^{n-1} x_{i+n}\right) - y\right\|$$

$\leq 2\varepsilon + M \|s(n;n) - y\|$ for every $n \geq N$ and $\ell \geq 0$.

Hence $\lim_{n\to\infty} s(n;2n+\ell) = y$ uniformly in $\ell \geq 0$ and so the conclusion follows from Lemma 8. Q.E.D.

Remark. The assumption " C is bounded " in Theorems 1 and 2 may be replaced by " $F(T) \neq \Phi$ ".

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References

[1] R.E.Bruck : A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces. Israel J. Math. 32, 107-116 (1979).

[2] R.E.Bruck : On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces. Israel J. Math. 38, 304-314 (1981).

[3] K.Goebel and W.A.Kirk : A fixed point theorem for asymptotically nonexpansive mappings. Proc. Amer. Math. Soc. 35, 171-174 (1972).

[4] N.Hirano : Nonlinear ergodic theorems and weak convergence theorems. J. Math. Soc. Japan 34, 35-46 (1982).

[5] N.Hirano and W.Takahashi : Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces. Kodai Math. J. 2, 11-25 (1979).

[6] K. Kobayasi and I. Miyadera : On the strong convergence of the Cesaro means of contractions in Banach spaces. Proc. Japan Acad. 56, 245-249 (1980).

[7] I. Miyadera and K. Kobayasi : On the asymptotic behaviour of almost-orbits of nonlinear contraction semigroups in Banach spaces. Nonlinear Analysis 6, 349-365 (1982).